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THE INVERSE GALOIS PROBLEM OVER FORMAL POWER SERIES FIELDS

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ABSTRACT

Consider a valuation ring R of a discrete Henselian field and a positive integer r. Let F be the quotient field of the ring $R[[X_1, \ldots, X_r]]$. We prove that every finite group occurs as a Galois group over F. In particular, if K_0 is an arbitrary field and $r \geq 2$, then every finite group occurs as a Galois group over $K_0((X_1, \ldots, X_r))$.

Introduction

The **inverse Galois problem** asks whether every finite group G occurs as a Galois group over the field $\mathbb{Q}$ of rational numbers. We then say that G is **realizable** over $\mathbb{Q}$. This problem goes back to Hilbert [Hil] who realized S_n and A_n over $\mathbb{Q}$. Many more groups have been realized over $\mathbb{Q}$ since 1892. For example, Shafarevich [Sha] finished in 1958 the work started by Scholz 1936 [Slz] and Reichardt 1937 [Rei] and realized all solvable groups over $\mathbb{Q}$. The last ten years have seen intensified efforts toward a positive solution of the problem. The area has become one of the frontiers of arithmetic geometry (see surveys of Matzat [Mat] and Serre [Se1]).

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Parallel to the effort of realizing groups over $\mathbb{Q}$, people have generalized the inverse Galois problem to other fields with good arithmetical properties. The most distinguished field where the problem has an affirmative solution is $\mathbb{C}(t)$. This is a consequence of the Riemann Existence Theorem from complex analysis.

Winfried Scharlau and Wulf-Dieter Geyer asked what is the absolute Galois group of the field of formal power series $F = K((X_1, \ldots, X_r))$ in $r \geq 2$ variables over an arbitrary field K. The full answer to this question is still out of reach. However, a theorem of Harbater (Proposition 1.3a) asserts that each Galois group is realizable over the field of rational function $F(T)$. By a theorem of Weissauer (Proposition 3.1), F is Hilbertian. So, G is realizable over F. Thus, the inverse Galois problem has an affirmative solution over F.

The goal of this note is to prove the same result in a more general setting.

THEOREM A: *Let R be the valuation ring of a discrete Henselian field K, let r be a positive integer, and let F be the quotient field of $R[[X_1, \ldots, X_r]]$. Then every finite group G is realizable over F.*

COROLLARY B:

(a) *Let K_0 be an arbitrary field and let $r \geq 2$. Then every finite group is realizable over $K_0((X_1, \ldots, X_r))$.*

(b) *Let $r \geq 1$ and let F be the quotient field of $\mathbb{Z}_p[[X_1, \ldots, X_r]]$ or of $\mathbb{Z}_{p,\mathrm{alg}}[[X_1, \ldots, X_r]]$. Then every finite group is realizable over F. Here $\mathbb{Z}_p$ is the ring of p-adic numbers and $\mathbb{Z}_{p,\mathrm{alg}}$ is the subring of all p-adic numbers which are algebraic over $\mathbb{Q}$.*

Proof: Apply Theorem A to $R = K_0[[X_1]]$, to $R = \mathbb{Z}_p$, and to $R = \mathbb{Z}_{p,\mathrm{alg}}$. ∎

The proof of Theorem A is a combination of several known results which we bring in this note.

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1. The theorem of Harbater and Liu

Let K be a field and let G be a finite group. We say that G is **regular** over K if there exists an absolutely irreducible polynomial $f \in K[T, X]$ which is Galois over $K(T)$ whose Galois group, namely, $\mathcal{G}(f(T, X), K(T))$ is isomorphic to G.

Alternatively, $K(T)$ has a Galois extension F which is regular over K such that $\mathcal{G}(F/K(T)) \cong G$.

We say that G is **regular over K with a rational point** if there exists a dominating Galois rational map of irreducible affine curves ϕ: $C \to \mathbb{A}^1$ defined over K such that C has a simple K-rational point and $\mathcal{G}(C/\mathbb{A}^1) \cong G$.

Remark 1.1: Base field extension. Note that if G is regular over a field K, then it is regular over every extension L of K. Indeed, we may take F as free from L and therefore as linearly disjoint from L over K [FrJ, Lemma 9.9].

Similarly, if G is regular over K with a rational point, then G is regular with a rational point over each extension of K. ∎

The condition on C to have a K-rational point implies that F is regular over K. Thus, "G is regular over K with a rational point" implies that "G is regular over K".

Indeed, let $E = K(T)$ be the function field of $\mathbb{A}^1$ and let F be the function field of C over K. By assumption, F/E is Galois with $\mathcal{G}(F/E) \cong G$. Also, there exists a place ϕ: $F \to K \cup \{\infty\}$ over K [JaR, Cor. A2]. It follows from the following well known lemma that F/K is regular.

LEMMA 1.2: *Let F/K be an extension of fields. If there exists a K-place ϕ: $F \to K \cup \{\infty\}$, then F/K is regular.*

Proof: Indeed, let $w_1, \ldots, w_n \in \tilde{K}$ be linearly independent over K and let $u_1, \ldots, u_n \in F$ such that $\sum_{i=1}^n u_i w_i = 0$ and not all u_i are 0. Assume without loss that $\phi(u_i/u_1) \in K$, $i = 1, \ldots, n$ and extend ϕ to a $\tilde{K}$-place $\tilde{\phi}$: $F\tilde{K} \to \tilde{K} \cup \{\infty\}$. Then apply $\tilde{\phi}$ to $\sum_{i=1}^n \frac{u_i}{u_1} w_i = 0$ to get the relation $\sum_{i=1}^n \phi(\frac{u_i}{u_1}) w_i = 0$. It follows that $1 = \phi(\frac{u_1}{u_1}) = 0$. This contradiction proves that F is linearly disjoint from $\tilde{K}$ over K. In other words, F/K is regular. ∎

Suppose now that K is an infinite field and that ϕ: $C \to \mathbb{A}^1$ is as above, with $C \subseteq \mathbb{A}^n$, $n \geq 2$. Then we may project C from an appropriate point of $\mathbb{A}^n(K)$ onto a curve $C' \subseteq \mathbb{A}^2$ such that C' is K-birationally equivalent to C and the K-rational simple point of C is mapped on a simple K-rational point of C'. Thus there exists an absolutely irreducible polynomial $f \in K[T, X]$ with $\mathcal{G}(f(T, X), K(T)) \cong G$ and there exists $a, b \in K$ such that $f(a, b) = 0$ and $\frac{\partial f}{\partial T}(a, b) \neq 0$ or $\frac{\partial f}{\partial X}(a, b) \neq 0$.

PROPOSITION 1.3: *Let R be local integral domain with a quotient field K such that $R \neq K$.*

(a) *(Harbater [Ha1, Thm. 2.3]) If R is complete, then each finite group is regular over K with a rational point.*

(b) *(Liu [Liu]) If R is a complete discrete valuation ring, then each finite group G is regular over K with a rational point.*

Remark 1.4: About the proofs of Harbater and Liu.

(a) Harbater uses 'mock covers' and 'Grothendieck's existence theorem' [GrD, (5.1.6)] in his proof. The rationality of the group over K is not explicitly stated in [Ha1, Thm. 2.3], but it can be deduced from the properties of the 'mock covers'.

(b) Liu [Liu] translates Harbater's method into 'rigid analytic geometry' for the case where R is a complete discrete valuation ring. We prove however, that this special case of Harbater's result implies the more general theorem. ∎

LEMMA 1.5: *Each complete local integral domain R which is not a field contains a complete discrete valuation ring.*

Proof: Let $\mathfrak{m}$ be the maximal ideal of R. Suppose first that $\mathrm{char}(R) = 0$. Then $\mathbb{Z} \subseteq R$ and there are two possibilities:

CASE A: $\mathbb{Z} \cap \mathfrak{m} \neq 0$. Then $\mathbb{Z} \cap \mathfrak{m} = p\mathbb{Z}$ for some prime number p. Since R is complete, $\mathbb{Z}_p \subseteq R$.

CASE B: $\mathbb{Z} \cap \mathfrak{m} = 0$. Since R is not a field, there exists $0 \neq x \in \mathfrak{m}$. If x were algebraic over $\mathbb{Q}$, then $a_n x^n + \cdots + a_1 x + a_0 = 0$ with $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $a_0 \neq 0$. But then $a_0 \in \mathbb{Z} \cap \mathfrak{m}$. This contradiction proves that x is transcendental over $\mathbb{Q}$. It follows that $\mathbb{Q}[x] \subseteq R$ and $\mathbb{Q}[x] \cap \mathfrak{m} = x\mathbb{Q}[x]$. The completion of $\mathbb{Q}[x]$ with respect to x is a discrete valuation ring which is contained in R.

Now suppose that $\mathrm{char}(R) = p$. Then $\mathbb{F}_p \cap \mathfrak{m} = 0$ and one continues as in Case B, replacing $\mathbb{Q}$ by $\mathbb{F}_p$. ∎

COROLLARY 1.6: *Proposition 1.3(b) implies Proposition 1.3(a).*

Proof: Let R be as in Proposition 1.3. Lemma 1.5 gives a complete valuation subring R_0 of R. By Proposition 1.3(b), G is regular over the quotient field of R_0 with a rational point. Hence G is also regular over K with a rational point. So, Proposition 1.3(a) is valid. ∎

2. Henselian fields

A field K is **defectless** with respect to a valuation v if each finite extension L of K satisfies

$$[L:K] = \sum_{w|v} e(w/v) f(w/v), \tag{1}$$

where w ranges over all valuations of L that extend v, $e(w/v)$ is the ramification index, and $f(w/v)$ is the relative residue degree of w/v. If (K, v) is Henselian, then v has a unique extension w to L. In this case we write $e(L/K)$ (resp., $f(L/K)$) instead of $e(w/v)$ (resp., $f(w/v)$). Then condition (1) simplifies to

$$[L:K] = e(L/K) f(L/K) \tag{2}$$

For example, each complete discrete valued field (K, v) is defectless [Rbn, p. 236].

LEMMA 2.1*: *Let (K, v) be a defectless Henselian discrete valued field, and let $(\hat{K}, \hat{v})$ be its completion. Then $\hat{K}/K$ is a regular extension.*

Proof: We have to prove that each finite extension L of K is linearly disjoint from $\hat{K}$ over K.

Indeed, as $\hat{K}/K$ is an immediate extension $e(\hat{K}/K) = 1$. Thus $e(\hat{K}L/\hat{K}) = e(\hat{K}L/K) = e(\hat{K}L/L)e(L/K) \geq e(L/K)$. Similarly the residue degree satisfy $f(\hat{K}L/\hat{K}) \geq f(L/K)$. Hence, by (2)

$$[\hat{K}L : \hat{K}] \leq [L:K] = e(L/K)f(L/K) \leq e(\hat{K}L/\hat{K})f(\hat{K}L/\hat{K}) = [\hat{K}L : \hat{K}].$$

Thus $[\hat{K}L : \hat{K}] = [L : K]$. Conclude that L is linearly disjoint from $\hat{K}$ over K. ∎

Suppose now that v is a discrete valuation of K (i.e., $v(K) = \mathbb{Z}$). Let O be its valuation ring, let L be a finite extension of K and let O' be the integral closure of K in L. If O' is a finitely generated O-module, then (1) holds [Se2, p. 26]. This is in particular the case if L/K is separable [Se2, p. 24]. Hence, if $\mathrm{char}(K) = 0$, then K is defectless with respect to v. If K is a function field of one variable over a field K_0, and v is a valuation of K which is trivial on K_0, then there exists a finitely generated ring R over K_0 and a prime ideal $\mathfrak{p}$ of R such that $R_{\mathfrak{p}}$ is the valuation ring of v. Since the integral closure of R in L is finitely generated as an R-module [La1, p. 120], the same holds for $R_{\mathfrak{p}}$. It follows that (K, v) is defectless.

* Lemmas 2.1 and 2.2 overlap with Lemma 2.13 and Corollary 2.14 of [Kul].

LEMMA 2.2: *Let (K, v) be a discrete Henselian valued field and let $(\hat{K}, \hat{v})$ be the completion of (K, v). Then (K, v) is defectless in each of the following cases:*

(a) $\mathrm{char}(K) = 0$.

(b) *(K, v) is the Henselization of a valued field (K_1, v_1), where K_1 is a function field of one variable over a field K_0 and v_1 is a valuation of K_1 which is trivial on K_0.*

Hence, by Lemma 2.1, in each of these cases, $\hat{K}/K$ is a regular extension.

Proof: By the paragraph that precedes the lemma, it suffices to consider only Case (b). Since (1) holds if L/K is separable, it suffices to prove (2) only in the case where L/K is a purely inseparable extension of degree q. Then there exists a finite extension K_2 of K_1 which is contained in K and a finite purely inseparable extension L_2 of K_2 of degree q such that $K \cap L_2 = K_2$ and $KL_2 = L$. Since K_2 is a function field of one variable over a finite extension of K_0, K_2 is defectless. Also $v_2 = v|_{K_2}$ has a unique extension w_2 to L_2. Hence, $e(w_2/v_2)f(w_2/v_2) = q$.

Denote now the unique extension of v to L by w. Then $w|_{L_2} = w_2$. Since (K, v) is also the Henselization of (K_2, v_2), we have $f(L/K) \geq f(w_2/v_2)$ (actually both degrees are 1) and $e(L/K) \geq e(w_2/v_2)$. So,

$$q = [L : K] \geq e(L/K)f(L/K) \geq e(w_2/v_2)f(w_2/v_2) = q$$

and therefore (2) holds, as desired. ∎

LEMMA 2.3: *Let (K, v) be a Henselian valued field and let $(\hat{K}, \hat{v})$ be its completion. Suppose that $\hat{K}/K$ is a regular extension. Then for each $0 \neq g \in K[X_1, \ldots, X_n]$ each point $\mathbf{x} \in (\hat{K})^n$ with $g(\mathbf{x}) \neq 0$ has a K-rational specialization $\mathbf{a}$ such that $g(\mathbf{a}) \neq 0$. Thus K is* **existentially closed** *in $\hat{K}$.*

Proof: Adding $g(\mathbf{x})^{-1}$ to $x_1, \ldots, x_n$ if necessary, we may assume that $g = 1$. By assumption, $K(\mathbf{x})$ is a separable extension of K. Let $u_1, \ldots, u_r$ be a separating transcendence base for $K(\mathbf{x})/K$ and let z be a primitive element for the finite separable extension $K(\mathbf{x})/K(\mathbf{u})$ which is integral over $K[\mathbf{u}]$. Then there exists an irreducible polynomial $f \in K[U_1, \ldots, U_r, Z]$ such that $f(\mathbf{u}, z) = 0$ and $f'(\mathbf{u}, z) \neq 0$ (the prime stands for derivative with respect to Z). Also, $x_i = h_i(\mathbf{u}, z)/h_0(\mathbf{u})$, for $h_i \in K[\mathbf{U}, Z]$ and $0 \neq h_0 \in K[\mathbf{U}]$.

Since (K, v) is dense in $(\hat{K}, \hat{v})$ we may approximate $u_1, \ldots, u_r, z$ by elements of K to any desired degree. Since K is Henselian, there exist $b_1, \ldots, b_r, c \in K$

such that $f(\mathbf{b}, c) = 0$ and $h_0(\mathbf{b}) \neq 0$. It follows that $(\mathbf{b}, c)$ is a K-specialization of $(\mathbf{u}, z)$.

Let now $a_i = h_i(\mathbf{b}, c)/h_0(\mathbf{b})$, $i = 1, \ldots, n$. Then $\mathbf{a}$ is a K-specialization of $\mathbf{x}$. ∎

LEMMA 2.4: *Let K be an existentially closed subfield of a field $\hat{K}$. If a finite group G is regular over $\hat{K}$ (resp., with a rational point), then G is also regular over K (resp., with a rational point).*

Proof: Suppose for example that G is regular over $\hat{K}$ with a rational point. Then, there exists an absolutely irreducible polynomial $f \in \hat{K}[T, X]$ which is Galois and monic in X such that $\mathcal{G}(f(T, X), \hat{K}(T)) \cong G$, and there exist $t, x \in \hat{K}$ such that $f(t, x) = 0$ and $\frac{\partial f}{\partial T}(t, x) \neq 0$ or $\frac{\partial f}{\partial X}(t, x) \neq 0$. Find $u_1, \ldots, u_n \in \hat{K}$ and a polynomial $g \in K(\mathbf{U})[T, X]$ such that $K[\mathbf{u}]$ is integrally closed, $g(\mathbf{u}, T, X) = f(T, X)$, $\mathcal{G}(g(\mathbf{u}, T, X), K(\mathbf{u}, T)) \cong G$, and there exist rational functions $p, q \in K(\mathbf{U})$ such that $t = p(\mathbf{u})$ and $x = q(\mathbf{u})$. By the Bertini-Noether theorem there exists $0 \neq h \in K(\mathbf{U})$ such that if a specialization $\mathbf{a}$ of $\mathbf{u}$ satisfies $h(\mathbf{a}) \neq 0$, then $g(\mathbf{a}, T, X)$ is well defined, Galois in X, and absolutely irreducible [FrJ, Prop. 9.29]. Also, $p(\mathbf{a})$ and $q(\mathbf{a})$ are well defined and $\frac{\partial f}{\partial T}(p(\mathbf{a}), q(\mathbf{a})) \neq 0$ or $\frac{\partial f}{\partial X}(p(\mathbf{a}), q(\mathbf{a})) \neq 0$. Choosing h such that the discriminant of $g(\mathbf{a}, T, X)$ with respect to X is nonzero, $\mathcal{G}(g(\mathbf{a}, T, X), K(\mathbf{a}))$ becomes isomorphic to a subgroup of $\mathcal{G}(g(\mathbf{u}, T, X), K(\mathbf{u}, T))$ [La2, p. 248, Prop. 15]. Since

$$
\begin{aligned}
|\mathcal{G}(g(\mathbf{a}, T, X), K(\mathbf{a}, T))| &= \deg_X g(\mathbf{a}, T, X) \\
&= \deg_X g(\mathbf{u}, T, X) = |\mathcal{G}(g(\mathbf{u}, T, X), K(\mathbf{u}, T))|,
\end{aligned}
$$

we have $\mathcal{G}(g(\mathbf{a}, T, X), K(\mathbf{a})) \cong G$. Since K is existentially closed in $\hat{K}$, we can choose $\mathbf{a}$ in K^n. Hence G is regular over K with a rational point.

Similarly one proves that if G is regular over $\hat{K}$, then it is also regular over K. ∎

THEOREM 2.5 (Florian Pop*): *Let (F, w) be a Henselian valued field. Then every finite group G is regular over F with a rational point.*

Proof: It is implicit in our assumptions that w is a nontrivial valuation.

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* Communicated to the author by Peter Roquette.

CLAIM: *(F,w) is an extension of a discrete Henselian valued field (K,v) which satisfies the conclusion of Lemma 2.3.*

Suppose first that $\mathrm{char}(F)=0$ and that w is nontrivial on $\mathbb{Q}$. Then $F_0=\tilde{\mathbb{Q}}\cap F$ is Henselian with respect to $w_0=w|_{F_0}$ [Jar, Cor. 11.2]. Hence, there exists p such that (F_0,w_0) is an extension of the Henselization $(\mathbb{Q}_{p,\mathrm{alg}},v_p)$ of $(\mathbb{Q},v_p)$, where v_p denotes the p-adic valuation. Let $K=\mathbb{Q}_{p,\mathrm{alg}}$ and $v=v_p$.

Next suppose that $\mathrm{char}(F)=0$ and that w is trivial on $\mathbb{Q}$. Then there exists $x\in F\smallsetminus\mathbb{Q}$ such that $w(x)\neq 0$. This element is transcendental over $\mathbb{Q}$. Thus w induces a nontrivial valuation v_0 on $\mathbb{Q}(x)$. Then $F_0=\widetilde{\mathbb{Q}(x)}\cap F$ contains the Henselization K of $\mathbb{Q}(x)$ with respect to v_0.

If $\mathrm{char}(F)=p$, then w is trivial on $\mathbb{F}_p$. Hence, as in the preceding paragraph, there exists $x\in F$ which is transcendental over $\mathbb{F}_p$ such that F contains a Henselization K of $\mathbb{F}_p(x)$.

In each case Lemma 2.2 asserts that (K,v) satisfies the conclusion of Lemma 2.3.

Let $\hat{K}$ be the completion of K with respect to v. By Proposition 1.3b, G is regular over $\hat{K}$ with a rational point. Hence, by Lemma 2.4, G is regular over F with a rational point. ∎

Recall that a field K is **PAC** if each nonempty absolutely irreducible variety which is defined over K has a K-rational point. Fried and Völklein [FV1] use complex analysis to prove that if K is a PAC field of characteristic 0, then each finite group G is regular over K. Völklein informed the author that the construction in [Voe] implies that G is even regular over K with a rational point. Pop has observed that the methods of this note imply the same result without any restriction on the characteristic:

THEOREM 2.6: *Let K be a PAC field and let G be a finite group. Then G is regular over K with a rational point.*

Proof: The field $\hat{K}=K((X))$ is regular over K, because the map $X\to 0$ extends to a place $\hat{K}\to K\cup\{\infty\}$ (Lemma 1.2). Since K is PAC this implies that K is existentially closed in $\hat{K}$ [FrJ, p. 139, Exer. 7]. By Proposition 1.3(b), G is regular over $\hat{K}$ with a rational point. Hence, by Lemma 2.4, G is regular also over K with a rational point. ∎

3. Hilbertian fields

An integral domain S with a quotient field F is a **Krull domain** if F has a family $\mathcal{V}$ of discrete valuations such that the intersection of their valuation rings is S and for each $0 \neq a \in K$ there are only finitely many $v \in \mathcal{V}$ such that $v(a) \neq 0$. For example, each Dedekind domain is a Krull domain. Also, if S is a Krull domain with a quotient field F, then the integral closure of S in any finite extension of F, the polynomial ring $S[X]$, and the ring of power series $S[[X]]$ are again Krull domains [Bou, pp. 487, 489, and 547].

The **dimension** of S is greater than 1, if S has a maximal ideal M which properly contains a nonzero prime ideal. ∎

PROPOSITION 3.1 (Weissauer [FrJ, Thm. 14.7]): *The quotient field of a Krull domain of dimension exceeding* 1 *is separably Hilbertian.*

Example 3.2: Ring of formal power series. Let R be either a field or a discrete valuation ring with maximal ideal $\mathfrak{m}$. Then, $S = R[[X_1, \ldots, X_r]]$ is a Krull domain. Indeed, it is even a unique factorization domain [Bou, p. 511].

Consider the ideal M of S which consists of all power series $\sum_{i=0}^{\infty} f_i$, where $f_i \in R[X_1, \ldots, X_r]$ is a form of degree i, $f_0 = 0$ if R is a field, and $f_0 \in \mathfrak{m}$ if R is a discrete valuation ring. Since $S/M \cong R$ if R is a field and $S/M \cong R/\mathfrak{m}$ if R is a discrete valuation ring, M is a maximal ideal. If R is a field (resp., discrete valuation ring) and $r \geq 2$ (resp., $r \geq 1$), then M contains the prime ideals generated by X_1 and by X_2 (resp., $\mathfrak{m}$ and by X_1) and neither of them is contained in the other. Hence $\dim(S) \geq 2$. It follows from Proposition 3.1 that the quotient field of S is separably Hilbertian. ∎

THEOREM A: *Let R be the valuation ring of a discrete Henselian field K, let r be a positive integer, and let F be the quotient field of $R[[X_1, \ldots, X_r]]$. Then every finite group G is realizable over F.*

Proof: Let G be a finite group. By Theorem 2.5, G is regular over the quotient field of R with a rational point. Hence, G is regular over F with a rational point. In particular, G is realizable over $F(T)$. By Example 3.2, F is separably Hilbertian. Hence G is realizable over F [FrJ, Lemma 12.12]. ∎

Remark 3.3: The case $r = 1$. By Puiseux's theorem, $G(\mathbb{C}((X))) \cong \hat{\mathbb{Z}}$. Hence, only cyclic groups can be realized over $\mathbb{C}((X))$. Thus, Corollary B(a) is false for $r = 1$. ∎

Remark 3.4: Cohomological dimension. We have already mentioned that every finite group is realizable over $\mathbb{C}(t)$. Moreover, the absolute Galois group, $G(\mathbb{C}(t))$, of $\mathbb{C}(t)$ is even a free profinite group of uncountable rank [Rib, p. 70]. In particular, $G(\mathbb{C}(t))$ is projective, that is, of cohomological dimension 1. On the other hand, use the notation of Theorem A and assume that there exists a prime $p \neq \mathrm{char}(K)$ such that $1 \leq \mathrm{cd}_p(G(K)) < \infty$. Then, as we explain in the next paragraph, $\mathrm{cd}_p(G(F)) \geq r+1$. In particular, although every group is realizable over F, not every embedding problem for $G(F)$ is solvable.

Indeed, let E be the quotient field of $R[[X_1, \ldots, X_{r-1}]]$. Induction on r gives, $\mathrm{cd}(G(E)) \geq r$. Hence, $\mathrm{cd}(G(E((X_r))) \geq r+1$ [Rib, p. 277]. Also, $E \subseteq E(X_r) \subseteq F \subseteq E((X_r))$. By Krasner's lemma [Jar, Prop. 12.3] $E(X_r)_s E((X_r)) = E((X_r))_s$ (L_s is the separable closure of a field L.) Hence $F_s E((X_r)) = E((X_r))_s$, and therefore, by Galois theory, $G(E((X_r)))$ is isomorphic to the closed subgroup $G(F_s \cap E((X_r))$ of $G(F)$. Conclude that $\mathrm{cd}(G(F)) \geq \mathrm{cd}_p(G(E((X_r))) \geq r+1$ [Rib, p. 204], as was to be shown. ∎

Denote the free profinite group of countable rank by $\hat{F}_\omega$.

Example 3.5: A field K over which every finite group is realizable but $\hat{F}_\omega$ is not realizable over K.

Let $G_1, G_2, G_3, \ldots$ be a listing of all finite groups. Consider the direct product $G = \prod_{i=1}^{\infty} G_i$. Then G is a profinite group of rank $\aleph_0$. Let $\phi\colon \tilde{G} \to G$ be the universal Frattini cover of G. Then $\tilde{G}$ is projective [FrJ, Prop. 20.33] of rank $\aleph_0$ [FrJ, Cor. 20.26]. Hence, there exists an algebraic extension K of $\mathbb{Q}$ which is PAC with $G(K) \cong \tilde{G}$ [FrJ, Thm. 20.22]. Then, each finite group is a quotient of $\tilde{G}$ and therefore it is realizable over K.

Assume now that $\hat{F}_\omega$ is realizable over K. Then, $\hat{F}_\omega$ is a quotient of $\tilde{G}$. It follows that there exists a Frattini cover ϕ of $\hat{F}_\omega$ onto a quotient $\bar{G}$ of G [FrJ, Lemma 20.35]. The kernel of ϕ is contained in the Frattini subgroup of $\hat{F}_\omega$ which is trivial [FrJ, Cor. 24.8]. Hence, $\hat{F}_\omega \cong \bar{G}$ and therefore there exists an epimorphism $\alpha\colon G \to \hat{F}_\omega$. But for each i, $\alpha(G_i)$ is a finite subgroup of $\hat{F}_\omega$. Since $\hat{F}_\omega$ is torsion free, $\alpha(G_i) = 1$. Since the G_i generate G, we obtain that $\hat{F}_\omega = \alpha(G) = 1$. This contradiction proves that $\hat{F}_\omega$ is not realizable over K.

Note that as K is PAC, the latter conclusion implies, in view of a result of Fried and Völklein [FV2, Thm. A], that K is not Hilbertian. So, our argument strengthens the one given in [FV2, Sect. 2, Example]. ∎

PROPOSITION 3.6 (W.-D. Geyer): *If K is an algebraically closed field of characteristic 0 and $r \geq 2$, then $\hat{F}_\omega$ is realizable over $K((X_1, \dots, X_r))$.*

Proof: Observe that $K(\frac{X_1}{X_2}) \subseteq K((X_1, \dots, X_r))$. As $t = \frac{X_1}{X_2}$ is transcendental over K, the absolute Galois group of $K(t)$ is free of rank which is equal to the cardinality of K [Rib, p. 70]. In particular $\hat{F}_\omega$ is a quotient of $G(K(t))$.* It follows from the next claim that $\hat{F}_\omega$ is realizable over $K((X_1, \dots, X_r))$.

CLAIM: *$K(t)$ is algebraically closed in $K((X_1, \dots, X_r))$.* Indeed, consider an algebraic element $f \in K((X_1, \dots, X_r))$ over $K(t)$. We prove that each prime divisor of $K(t)$ is unramified in $K(t, f)$. It will follow that $f \in K(t)$, [FrJ, Prop. 2.15], as desired.

To this end consider $c \in K$ and let $u \doteq t - c$. Then $X_1 = X_2(u + c)$ and therefore

$$K(u) = K(t) \subseteq K((X_1, X_2, \dots, X_r)) \subseteq K((u, X_2, \dots, X_r)) \subseteq K((u))((X_2, \dots, X_r)) = F.$$

The map $X_i \mapsto 0$, $i = 2, \dots, r$, extends to a $K((u))$-place $\phi\colon F \to K((u)) \cup \{\infty\}$ which extends further to a place $\phi\colon \tilde{F} \to \widetilde{K((u))} \cup \{\infty\}$ which fixes each element of $\widetilde{K((u))}$. In particular, as $f \in \widetilde{K(u)} \cap F$, we have $f = \phi(f) \in K((u))$. But $K((u))/K(t)$ is unramified at the zero $(t-c)_0$ of $t - c$. So, $(t-c)_0$ is unramified in $K(t, f)$. Finally, replace t by $\frac{X_2}{X_1}$ to conclude that also $(t)_\infty$ is unramified in $K(t, f)$, as desired. ∎

Example 3.5 and Proposition 3.6 naturally raise the following question:

PROBLEM 3.7: *Let K be an arbitrary field and let $r \geq 2$. Is $\hat{F}_\omega$ realizable over $K((X_1, \dots, X_r))$?*

Remark 3.8: Harbater [Ha2, Prop. 2.3] proves that if O is the ring of integers of a number field K and F is the quotient field of $O[[X]]$, then every finite group G is realizable over F. Moreover, F has a Galois extension $\hat{F}$ which is regular over K such that $\mathcal{G}(\hat{F}/F) \cong G$. Note that as O is a Dedekind domain, $O[[X]]$ is a Krull domain of dimension at least 2. Hence, by Proposition 3.1, F is Hilbertian. ∎

In view of Theorem A and Remarks 3.3 and 3.8 we may ask:

* Florian Pop has recently announced a '$\frac{1}{2}$ Riemann existence theorem' from which the same result follows also if $\mathrm{char}(K) \neq 0$. If we use Pop's theorem, then Proposition 3.6 will hold for an arbitrary algebraically closed field.

PROBLEM 3.8: *Let O be a domain of characteristic* 0 *which is not a field. Denote the quotient field of $O[[X]]$ by F. Is every finite group realizable over F?* ∎

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